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# JØRGENSEN'S INEQUALITY FOR COMPLEX HYPERBOLIC 2- SPACE

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## 1 Introduction

Jørgensen's inequality gives a necessary condition for non-elementary two generator group of isometries of hyperbolic space to be discrete. We give analogues of Jørgensen's inequality for non-elementary groups of isometries of complex hyperbolic 2-space generated by two elements, one of which is either loxodromic or boundary elliptic.

This is a joint work with Jiang Yueping (Hunan University) and John. R. Parker (University of Durham).

## 2 The classical Jørgensen's inequality

We discuss the original inequality of Jørgensen and reformulate in a way that we can generalize. Jørgensen takes two elements  $A$  and  $B$  in  $SL(2, \mathbb{C})$  and says that if

$$|tr^2(A) - 4| + |tr(ABA^{-1}B^{-1}) - 2| < 1,$$

then the group  $\langle A, B \rangle$  generated by  $A$  and  $B$  is either elementary or not discrete. In this paper we will only be concerned with the cases where  $A$  is loxodromic or elliptic. We may reformulate Jørgensen's inequality in terms of cross ratios of fixed points. Jørgensen's inequality is equivalent to

*Theorem 1. Let  $A$  and  $B$  be elements of  $SL(2, \mathbb{C})$  so that  $A$  is either loxodromic or elliptic with fixed points  $\mu$  and  $\nu$  in  $\hat{\mathbb{C}}$ . Let  $M = |tr^2(A) - 4|^{\frac{1}{2}}$ . If either*

$$M^2(|[B(\mu), \nu, \mu, B(\nu)]| + 1) < 1 \quad \text{or} \quad M^2(|[B(\mu), \mu, \nu, B(\nu)]| + 1) < 1,$$

*then the group  $\langle A, B \rangle$  is either elementary or not discrete.*

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### 3 Preliminaries

Let  $\mathbb{C}^{2,1}$  be a complex vector space of dimension 3 with the Hermitian form of signature (2,1). We choose the Hermitian form on  $\mathbb{C}^{2,1}$  to be given by the matrix  $J$

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus  $\langle z, w \rangle = w^* J z = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1$ .

We define the Siegel domain model of complex 2-space,  $\mathbb{H}_{\mathbb{C}}^2$  as follows: We identify points of  $\mathbb{H}_{\mathbb{C}}^2$  with their horospherical coordinates,  $z = (\zeta, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ = \mathbb{H}_{\mathbb{C}}^2$ . Similarly, points in  $\partial \mathbb{H}_{\mathbb{C}}^2 = \mathbb{C} \times \mathbb{R} \cup \{\infty\}$  are either  $z = (\zeta, v, 0) \in \mathbb{C} \times \mathbb{R} \times \{0\}$  or a point at infinity, which is denoted by  $\infty$ . Define the map  $\phi : \overline{\mathbb{H}_{\mathbb{C}}^2} \rightarrow \mathbb{P}\mathbb{C}^{2,1}$  by

$$\begin{aligned} \phi &: (\zeta, v, u) \mapsto [(-|\zeta|^2 - u + iv)/2, \zeta, 1]^t, \\ \phi &: \infty \mapsto [1, 0, 0]^t. \end{aligned}$$

The map  $\phi$  is a homeomorphism from  $\mathbb{H}_{\mathbb{C}}^2$  to the set of points  $z$  in  $\mathbb{P}\mathbb{C}^{2,1}$  with  $\langle z, z \rangle < 0$ . Also  $\phi$  is a homeomorphism from  $\partial \mathbb{H}_{\mathbb{C}}^2$  to the set of points  $z$  in  $\mathbb{P}\mathbb{C}^{2,1}$  with  $\langle z, z \rangle = 0$ . Let  $L$  be a complex line intersecting  $\mathbb{H}_{\mathbb{C}}^2$ . Then  $\phi(L)$  is a 2-dimensional subspace of  $\mathbb{C}^{2,1}$ . The orthogonal complement of this space is a one (complex) dimensional subspace of  $\mathbb{C}^{2,1}$  spanned by a vector  $p$  with  $\langle p, p \rangle > 0$ . Without loss of generality, we take  $\langle p, p \rangle = 1$  and call  $p$  the polar vector corresponding to the complex line  $L$ .

The Bergman metric on  $\mathbb{H}_{\mathbb{C}}^2$  is defined by the following formula for the distance  $\rho$  between points  $z$  and  $w$  of  $\mathbb{H}_{\mathbb{C}}^2$ :

$$\cosh\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle \phi(z), \phi(w) \rangle \langle \phi(w), \phi(z) \rangle}{\langle \phi(z), \phi(z) \rangle \langle \phi(w), \phi(w) \rangle}.$$

The holomorphic isometry group of  $\mathbb{H}_{\mathbb{C}}^2$  with respect to the Bergman metric is the projective unitary group  $PU(2, 1)$  and acts on  $\mathbb{P}\mathbb{C}^{2,1}$  by matrix multiplication. A matrix  $g \in GL(3, \mathbb{C})$  is in  $PU(2, 1)$  if and only if it preserves the Hermitian form given by  $J$ . For four distinct points  $z_1, z_2, w_1, w_2$  of  $\overline{\mathbb{H}_{\mathbb{C}}^2}$  the cross-ratio is defined as

$$|[z_1, z_2, w_1, w_2]| = \left| \frac{\langle \phi(w_1), \phi(z_1) \rangle \langle \phi(w_2), \phi(z_2) \rangle}{\langle \phi(w_2), \phi(z_1) \rangle \langle \phi(w_1), \phi(z_2) \rangle} \right|.$$

In order to represent the holomorphic isometries of  $\mathbb{H}_{\mathbb{C}}^2$ , we work with the special unitary group  $SU(2, 1)$  throughout this paper.

## 4 Subgroups with loxodromic generators

We give our results about the subgroups with loxodromic elements. Let  $A$  be a loxodromic element with fixed points  $\mu$  and  $\nu$  in  $\partial\mathbf{H}_{\mathbb{C}}^2$ . Suppose that  $A$  has a complex dilation factor  $\lambda(A)$ . We define a conjugation invariant factor  $M$  by

$$M = |\lambda(A) - 1| + |\lambda(A)^{-1} - 1|.$$

**Theorem 2.** *Let  $A$  be a loxodromic element of  $SU(2, 1)$  fixing  $\mu$  and  $\nu$ , and let  $B$  be any element of  $SU(2, 1)$ . If either*

$$M(|[B(\mu), \nu, \mu, B(\nu)]|^{1/2} + 1) < 1 \quad \text{or} \quad M(|[B(\mu), \mu, \nu, B(\nu)]|^{1/2} + 1) < 1,$$

*then the group  $\langle A, B \rangle$  is either elementary or not discrete.*

**Theorem 3.** *Let  $A$  be a loxodromic element of  $SU(2, 1)$  fixing  $\mu$  and  $\nu$ , and let  $B$  be any element of  $SU(2, 1)$ . If  $M \leq \sqrt{2} - 1$  and*

$$|[B(\mu), \nu, \mu, B(\nu)]| + |[B(\mu), \mu, \nu, B(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

*then the group  $\langle A, B \rangle$  is either elementary or not discrete.*

We can show that neither theorem is a consequence of the other one.

## 5 Subgroups with boundary elliptic elements

Let  $A$  be a boundary elliptic element of  $SU(2, 1)$ . Then  $A$  fixes a complex line in  $\mathbf{H}_{\mathbb{C}}^2$ . We denote this complex line by  $L_A$  and its polar vector by  $p_A$ . The fixed complex line of  $BAB^{-1}$  is  $B(L_A)$ , which has the polar vector  $B(p_A)$ . Normalizing  $p_A$  and  $B(p_A)$  so that  $\langle p_A, p_A \rangle = \langle B(p_A), B(p_A) \rangle = 1$ , we have three cases:

(1) If  $|\langle p_A, B(p_A) \rangle| < 1$ , then  $L_A$  and  $B(L_A)$  intersect at a point in  $\mathbf{H}_{\mathbb{C}}^2$ . Moreover,  $|\langle p_A, B(p_A) \rangle| = \cos \psi$ , where  $\psi$  is the angle of intersection between  $L_A$  and  $B(L_A)$ . In particular, if  $|\langle p_A, B(p_A) \rangle| = 0$ , then  $L_A$  and  $B(L_A)$  intersect orthogonally.

(2) If  $|\langle p_A, B(p_A) \rangle| = 1$ , then either  $B(L_A) = L_A$  or  $L_A$  and  $B(L_A)$  are asymptotic at a point in  $\partial\mathbf{H}_{\mathbb{C}}^2$ .

(3) If  $|\langle p_A, B(p_A) \rangle| > 1$ , then  $L_A$  and  $B(L_A)$  are ultraparallel, that is, they are disjoint and have a common orthogonal complex geodesic. Moreover,  $|\langle p_A, B(p_A) \rangle| = \cosh \frac{\rho}{2}$ , where  $\rho$  is the distance between  $L_A$  and  $B(L_A)$ .

For a boundary elliptic element  $A \in SU(2, 1)$  we define the order of  $A$  as

$$\text{ord}(A) = \inf\{m > 0 : A^m = I\}.$$

Theorem 4. Let  $A$  be a boundary elliptic element of  $SU(2, 1)$  which rotates through an angle  $\theta = 2\pi/n$  about a complex line  $L_A$ . Let  $B$  be any element of  $SU(2, 1)$  so that  $B(L_A) \neq L_A$ . If one of the following three conditions (1), (2) and (3) is satisfied, then the group  $\langle A, B \rangle$  is not discrete.

- (1)  $L_A$  and  $B(L_A)$  intersect at an angle  $\psi \neq \pi/2$  and  $\text{ord}(A) = n \geq 6$ .
- (2)  $L_A$  and  $B(L_A)$  are asymptotic and  $\text{ord}(A) = n \geq 7$ .
- (3)  $L_A$  and  $B(L_A)$  are ultraparallel and

$$\left| \cosh \frac{\rho}{2} \sin \frac{\theta}{2} \right| < \frac{1}{2},$$

where  $\rho$  is the distance between  $L_A$  and  $B(L_A)$ .

If  $L_A$  and  $B(L_A)$  intersect orthogonally and

$$|\text{tr}(B) \sin \frac{\theta}{2}| < \frac{1}{2},$$

then the group  $\langle A, B \rangle$  is either elementary or not discrete.

Theorem 5. Let  $A$  be a boundary elliptic element fixing the complex line  $L_A$  spanned by  $\mu$  and  $\nu$  in  $\partial H_{\mathbb{C}}^2$ . Suppose that  $B$  is any element of  $SU(2, 1)$  for which  $L_A$  and  $B(L_A)$  do not intersect orthogonally. If either

$$M(|[B(\mu), \nu, \mu, B(\nu)]|^{1/2} + 1) < 1 \quad \text{or} \quad M(|[B(\mu), \mu, \nu, B(\nu)]|^{1/2} + 1) < 1,$$

then the group  $\langle A, B \rangle$  is either elementary or not discrete.

Theorem 6. Let  $A$  be a boundary elliptic element fixing the complex line  $L_A$  spanned by  $\mu$  and  $\nu$  in  $\partial H_{\mathbb{C}}^2$ . Suppose that  $B$  is any element of  $SU(2, 1)$  for which  $L_A$  and  $B(L_A)$  do not intersect orthogonally. If  $M \leq \sqrt{2} - 1$  and

$$|[B(\mu), \nu, \mu, B(\nu)]| + |[B(\mu), \mu, \nu, B(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group  $\langle A, B \rangle$  is either elementary or not discrete.

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